IMO Training: Basic Proof Techniques

Hints

How to use this sheet. A hint is there to help you take one more step, not to give the argument away. Before reading one, write down what you tried and where you got stuck. After reading it, put the sheet away and go back to the problem. The numbers match the problem sheet.

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1. With n people the possible counts are $0, 1, \ldots, n-1$. Can the counts 0 and $n-1$ both occur?

2. What are the smallest and the largest possible digit sums of a three-digit number?

3. Look at the parities of the two coordinates. How many parity types are there?

4. Cut the board into four 1×1 squares. To show that fewer darts are not enough, use the corners.

5. Join the midpoints of the three sides.

6. Group the numbers into the pairs $\{1, 2\}, \{3, 4\}, \ldots, \{2n-1, 2n\}$.

7. Look at the remainders on division by n.

8. Count the pairs (member, committee that member serves on) in two ways.

9. Add up, city by city, the number of roads leaving it. How often is each road counted?

10. Add up all the handshake counts. What parity must the total have?

11. How does the number of pieces change with each break?

12. Use the previous problem. Do the players have any real choice?

13. Play piles of $1, 2, \ldots, 7$ stones. Which pile sizes do you want to leave your opponent?

14. Tabulate winning and losing piles for $n = 0, 1, \ldots, 12$. Note that 4 and 1 leave the same remainder on division by 3.

15. What would happen if the two piles were equal and it was your opponent's turn?

16. Consider the running totals $S_0 = 0,\ S_1 = a_1,\ S_2 = a_1 + a_2,\ \ldots,\ S_n$ and their remainders on division by n.

17. Pair up the numbers of the list that sum to 104. Which numbers have no partner? How many boxes is that?

18. Which pairs of remainders on division by 8 give a sum or a difference divisible by 8? Group the remainders into boxes accordingly.

19. Look at remainders on division by 3. Either all three remainders occur, or some remainder occurs at least three times.

20. Write each number as 2^k times an odd number. How many different odd parts are possible?

21. How many nonempty subsets do the ten numbers have, and how many different sums are possible? Afterwards, remove the common elements.

22. How many different colorings can a single column of three squares have?

23. Locker n is touched once for each divisor of n. For which n is the number of divisors odd?

24. What does one move do to the parity of the number of tails?

25. Track the parity of the number of red marbles.

26. Look at the differences between the three counts, modulo 3.

27. Compute the net change in the number of heads for each of the four spells. What do the four numbers have in common?

28. What does one domino cover in the usual chessboard coloring?

29. The chessboard coloring gives nothing. Label each square by the parity of its row and the parity of its column: four labels. How many squares of each label does one tile cover?

30. In the chessboard coloring, how many black squares can one T-tetromino cover? And 25 of them?

31. Use three colors along the diagonals: color the square in row i, column j by the remainder of $i + j$ on division by 3.

32. Color the cubes like a three-dimensional chessboard. How many cubes are there of each color, and which colors are the corner and the center?

33. Color the grid like a chessboard. How many squares are there of each color?

34. $|a - b|$ has the same parity as $a + b$. What happens to the parity of the sum of all numbers on the board?

35. Look at the reciprocal of the new number.

36. Add 1 to the new number and factor.

37. A grasshopper at a jumping over one at b lands at $2b - a$. What does that do to its parity?

38. What does a jump do to the parity of each coordinate of the jumping grasshopper?

39. What does turning over a row or a column of eight coins do to the parity of the number of tails?

40. Look at the perimeter of the infected region. Can it grow when a new square becomes infected?

41. Try a diagonal.

42. Count the unit faces that separate an infected cube from a healthy cube or from the outside. Compare with Problem 40.

43. Count the pairs (room, door of that room), room by room and door by door.

44. Add up the number of edges of each face. How often is each edge counted?

45. Pick one person. Among the other five, at least three stand in the same relation to that person. Now look at those three.

46. Pick one person. The other 16 are split into three kinds, so at least six are of one kind. Then use Problem 45.

47. Play the small piles $n = 1, 2, \ldots, 8$ and write down the losing ones. What do they have in common?

48. Where should the first coin go so that every later move of the opponent has a mirror image?

49. After the first move, the second player can leave two separate arcs of equal length. What then?

50. A number is divisible by 11 exactly when the alternating sum of its digits is. How can the second player make each pair of neighboring boxes cancel?

51. You are allowed to turn coins over. Set aside ten coins. If that pile has h heads, how many heads are in the other pile? Now turn all ten over.

52. From a distance, two ants that meet and turn around look exactly like two ants that walk straight through each other.

53. Suppose the set has at least two elements. Take two that are neighbors when the set is listed in increasing order.

54. Among the marked cells, look at the highest row, and in it the leftmost cell.

55. Fix two of the points. Choose the third by a measurement that makes "another point inside the triangle" impossible.

56. Suppose there were a cycle and look at its longest edge. Who drew it?

57. The two closest people throw at each other. If anyone else throws at one of them, count balloons. If not, set the pair aside and repeat.

58. Take a longest path in the tree and look at its two ends.

59. Take a longest path. Its endpoint has a second neighbor. Where can that neighbor be?

60. Take a player with the most wins.

61. Use induction on the number of players. Where in the line can the new player be inserted?

62. Choose the committee first and then the chair; or choose the chair first and then the rest.

63. A split into piles of sizes A and B separates exactly AB pairs of stones. How often is each pair of stones separated in the whole process?

64. Count the pairs of students inside each club. Can the same pair occur in two clubs?

65. Use $w_i + \ell_i = n - 1$ and factor $w_i^2 - \ell_i^2$. What is the total number of wins compared with the total number of losses?

66. Make a table with a tick in row i, column k whenever i divides k. Count the ticks by columns and by rows.

67. Fix one element x of S. Pair each subset with the subset obtained by adding x if it is absent and removing it if it is present.

68. What is left over when you choose k elements?

69. Write the subset as $a_1 < a_2 < \cdots < a_k$ and push the elements together: replace a_i by $a_i - (i-1)$. What kind of set do you get?

70. Horizontal dominoes come in stacked pairs covering two neighboring columns i and $i+1$. Record the numbers i.

71. Draw a partition as rows of dots, the longest row on top. Now read the same picture column by column.

72. Consider the remainders of $2^0, 2^1, 2^2, \ldots$ on division by 7777. How many powers do you need before two remainders coincide?

73. Two of the powers $3, 3^2, 3^3, \ldots$ leave the same remainder on division by 10000. Then cancel a common factor, and check that you are allowed to.

74. Look at pairs of consecutive Fibonacci numbers modulo 10000, starting from $(0, 1)$. Some pair repeats, and the rule that produces the next pair can be run backwards.

75. Find a closed tour of all 64 squares that always steps to a neighboring square. Removing two squares of opposite colors cuts the tour into one or two paths. Why does each contain an even number of squares?

76. Induction on n. Cut the board into four quarters; where should the first tromino go?

77. Sort the lengths. If no three form a triangle, each length is at least the sum of the two before it. How fast must the lengths grow?

78. One direction: walk around a cycle and watch the teams. Other direction: pick one person in each connected group and sort everyone by the parity of their distance from that person.

79. Write the numbers in a line from any starting point and look at the running totals. Where is the smallest one? Start just after it.

80. Let the cop always step toward the robber. Look at the part of the tree the robber can still reach without passing the cop.

81. Start from any division and count the pairs of enemies in different rooms. If someone is unhappy, move them. What happens to the count?

82. Among all divisions into two groups, take one with the most friendships between the groups. What if someone had more friends in their own group than in the other?

83. Among all ways of pairing the points, take one with the smallest total length. What if two segments crossed?

84. Count triples consisting of a pair of students and a common friend of both. Compare the total with the number of pairs.

85. If two people do not know each other, how many acquaintances must they have in common among the other $2n - 2$?

86. Color the positions in the row alternately. What can the first player always do, and what does that force the second player to do?

87. Describe a position by the lengths of its two rows and find the losing positions among small cases.

88. Split $1, \ldots, 100$ into chains by their remainder on division by 10. How many numbers can you take from one chain without two differing by 10? For 11, build a set out of blocks of consecutive numbers.

89. In each column some two squares share a color; record which two rows and which color. For the other direction, arrange columns so that no record repeats.

90. A column with r red squares contains $\binom{r}{2} + \binom{8-r}{2}$ same-colored pairs. How small can that be? Compare the total over eight columns with the number of possible records (pair of rows, color).

91. Let S_i be the number of games played up to day i. Put $S_1, \ldots, S_{77}$ and $S_1 + 21, \ldots, S_{77} + 21$ together. How large can these numbers be?

92. Averaging over all 100 triples gives only 151.5. Leave one number out and average over fewer triples that do not overlap.

93. For each entry record two numbers: the length of the longest increasing subsequence ending there, and of the longest decreasing one (a subsequence keeps the left-to-right order but may skip entries). If both are always at most 3, how many different records are possible?

94. Count all paths first. A bad path touches the line $y = x + 1$; reflect the part of the path after its first touch in that line. Where does the reflected path end?

95. Put all the even numbers first and all the odd numbers after them. How should each half be ordered? Use induction.

96. Every positive integer is 2^a times an odd number in exactly one way, and every positive integer is a sum of distinct powers of 2 in exactly one way.

97. Label the squares $0, 1, \ldots, 63$ in binary. Look for a way to combine the labels of all squares showing heads into one label, such that turning over a single coin can change the combined label to anything you like.

98. Suppose there is no triangle and take a vertex v of largest degree d. Can two neighbors of v be joined? So where does every edge have an endpoint?

99. Suppose not all points are on one line. Among all pairs (point, line through two of the points) with the point off the line, take one with the smallest distance.

100. If a number occupies r rows and c columns, then $rc \geq 100$, hence $r + c \geq 20$. Add this up over all hundred numbers.