

IMO Training: Basic Proof Techniques

A Problem-Solving Curriculum

Instructions. This sheet is designed as general preparation for Olympiad problem solving. The focus is on basic proof techniques that appear throughout olympiad mathematics: finding invariants, using parity and modular arguments, applying the pigeonhole principle, counting the same object in two ways, choosing an extremal example, and reducing a problem to a simpler structure.

1. ★ At a party with at least two people, prove that two people have shaken hands with the same number of other people.
2. ★ Among any 28 three-digit positive integers, prove that two have the same sum of digits.
3. ★ Place any 5 points in the plane with integer coordinates. Prove that the midpoint of some pair of them also has integer coordinates.
4. ★ What is the smallest number of darts one must throw at a 2×2 square board to guarantee that two darts are at distance at most $\sqrt{2}$?
5. ★ Place 5 points inside an equilateral triangle of side length 2. Prove that two of them are at distance at most 1.
6. ★ Choose any $n + 1$ distinct integers from $\{1, 2, \dots, 2n\}$. Prove that two of the chosen integers are relatively prime.
7. ★ Let n be a positive integer. Prove that among any $n + 1$ integers, two have a difference divisible by n .
8. ★ In a parliament with 100 members, every member serves on exactly 4 committees, and every committee has exactly 5 members. How many committees are there?
9. ★ The Kingdom of Graphia has 15 cities. The king wants every city to be connected by direct roads to exactly 3 other cities. Prove that this is impossible.
10. ★ At a party with 23 people, some pairs shook hands. Prove that someone shook hands with an even number of people (zero is even).
11. ★ An $m \times n$ chocolate bar is divided into unit squares. In one move, one may break one existing piece along a grid line. How many breaks are needed to separate the whole bar into unit squares?
12. ★ A 6×8 chocolate bar is broken into unit squares. Two players alternately break one existing piece along a grid line. The player who makes the last break wins. Who wins?
13. ★ A pile contains 100 stones. Two players alternately remove either 1 or 2 stones. The player who removes the last stone wins. Which player has a winning strategy?
14. ★★ A pile contains n stones. A move removes 1, 2, or 4 stones. The player who takes the last stone wins. Which pile sizes are losing positions?
15. ★ Two piles contain 10 and 12 stones. A move removes any positive number of stones from exactly one pile. The player who takes the last stone wins. Who has a winning strategy?

16. **★★** Given any sequence of n positive integers, prove that some consecutive block has sum divisible by n .
17. **★★** Choose 19 distinct numbers from $1, 4, 7, \dots, 100$. Prove that two of them sum to 104.
18. **★★** Given any six integers, prove that two of them have a sum or a difference divisible by 8.
19. **★★** Given any five integers, prove that three of them have a sum divisible by 3.
20. **★★** Prove that every subset of $\{1, 2, \dots, 2026\}$ with 1014 elements contains two numbers a, b such that $a \mid b$.
21. **★★** Write down any 10 distinct two-digit numbers. Prove that one can find two disjoint nonempty groups of these numbers with the same sum.
22. **★★** Color every square of a 3×9 grid red or blue. Prove that there is a rectangle whose four corner squares all have the same color.
23. **★★** One hundred lockers are closed and numbered $1, 2, \dots, 100$. Student 1 changes the state of every locker; student 2 changes the state of every locker numbered by a multiple of 2; and so on, up to student 100, who changes the state of locker 100. Which lockers are open at the end?
24. **★★** There are 100 coins in a circle, all showing heads. In one move, one may flip any 4 consecutive coins. Can one reach a position with exactly one tail?
25. **★** A jar contains 5 red and 6 green marbles. Repeatedly remove two marbles and replace them as follows: two green become one green; one of each color becomes one red; two red become one green. Continue until one marble remains. What color is it?
26. **★★** There are 13 red, 15 green, and 17 blue chameleons. Whenever two chameleons of different colors meet, they both change into the third color. Can all chameleons eventually become the same color?
27. **★★** A magical dragon has 100 heads. A wizard can cast four types of spells. A spell makes 15, 17, 20, or 5 heads disappear, respectively, and then causes 24, 2, 14, or 17 new heads to grow. Can the wizard make all the dragon's heads disappear?
28. **★** An 8×8 chessboard has two opposite corner squares removed. Can the remaining board be tiled by dominoes?
29. **★★** Prove that a 10×10 board cannot be tiled by twenty-five 1×4 rectangles, each placed horizontally or vertically.
30. **★★** A standard 10×10 board is to be tiled by 25 T-shaped tetrominoes. Is this possible?
31. **★★** Take an 8×8 chessboard and remove the top-left corner square. Can the remaining 63 squares be tiled by twenty-one 1×3 straight tiles?
32. **★★** A mouse starts at one corner cube of a $3 \times 3 \times 3$ block of cheese. It eats one cube at a time and moves only to a face-adjacent cube. Can it eat all 27 cubes and finish at the center cube?
33. **★★** There is one bug on each square of a 7×7 grid. At the same time, every bug crawls to an orthogonally adjacent square. Prove that after the move, at least one square is empty.
34. **★★** Let n be odd. The numbers $1, 2, \dots, 2n$ are on a board. Repeatedly erase two numbers a, b and write $|a - b|$, until one number remains. Prove that the final number is odd.

35. $\star\star$ The numbers $1, 2, \dots, 100$ are on a board. At each step, erase two numbers a, b and write $\frac{ab}{a+b}$. Prove that the final number is independent of the choices, and find it.
36. $\star\star\star$ The numbers $1, 2, \dots, 100$ are written on a blackboard. At each step, erase two numbers a, b and replace them by $a + b + ab$. Prove that the final number is independent of the choices and find it.
37. $\star$ Three grasshoppers sit on a number line at $0, 1, 3$. In a move, one jumps over another and lands the same distance beyond it. Can they ever occupy $0, 1, 2$?
38. $\star\star$ Three grasshoppers sit on an infinite grid at $(0, 0), (1, 0), (0, 1)$. A move consists of one grasshopper jumping over another and landing the same distance away on the other side. Can they reach $(0, 0), (1, 0), (1, 1)$?
39. $\star\star$ An 8×8 grid of coins has exactly one tail and 63 heads. In one move, one may flip every coin in a row or every coin in a column. Is it possible to reach all heads?
40. $\star\star$ An epidemic breaks out on a 10×10 grid. Initially exactly 9 squares are infected. A healthy square becomes infected if it shares an edge with at least two infected squares. Can the whole grid become infected?
41. $\star\star$ Same rules as the previous problem: a healthy square of the 10×10 grid becomes infected if it shares an edge with at least two infected squares. Show that 10 initially infected squares can infect the whole grid.
42. $\star\star\star$ A $5 \times 5 \times 5$ block consists of unit cubes. A healthy cube becomes infected when at least three face-neighbors are infected. Prove that fewer than 25 initially infected cubes can never infect the whole block.
43. $\star\star$ In a house, every room has an even number of doors. Some doors lead from one room to another; the others lead outside. Prove that the number of doors leading outside is even.
44. $\star\star$ A polyhedron is a solid whose surface is made of flat polygons, called faces, such as a cube or a pyramid; every edge of the surface belongs to exactly two faces. Is there a polyhedron with an odd number of faces, each of which has an odd number of edges?
45. $\star\star$ Among any 6 people, prove that there are either 3 mutual friends or 3 mutual strangers.
46. $\star\star\star$ At a party of 17 people, every pair are either mutual friends, mutual enemies, or mutual strangers. Prove that there are 3 people who are all mutual friends, all mutual enemies, or all mutual strangers.
47. $\star\star$ A pile contains n stones. In one move, a player may remove a positive number of stones, but not more than half the pile. The player who cannot move loses. Determine the losing positions.
48. $\star\star$ Coins of equal size are placed on a round table without overlapping. Two players alternately place one coin; whoever cannot place a coin loses. Prove that the first player can force a win.
49. $\star\star$ Ten coins lie in a circle. Players alternately remove either one coin or two adjacent coins, where two coins count as adjacent only if no coin has been removed from between them. The player who takes the last coin wins. Who has a winning strategy?
50. $\star\star$ Two players alternately write digits in 20 boxes from left to right, producing a string of 20 digits. The second player wins if the number it represents is divisible by 11. Prove that the second player can force a win.

51. **★★** One hundred coins lie on a table, exactly 10 showing heads. Blindfolded, you cannot tell which side is up. Split the coins into two piles with the same number of heads.
52. **★★** One hundred ants stand at distinct points on a one-meter stick. Each walks left or right at 1 cm per second. When two ants meet, both turn around; an ant reaching an end falls off. Prove that all ants have fallen off within 100 seconds.
53. **★** A finite set of real numbers has the property that the average of any two distinct numbers in the set also belongs to the set. Prove that the set has at most one element.
54. **★** A nonempty finite set of cells in an infinite square grid is marked. Prove that some marked cell has at most two marked neighbors, where neighbors share an edge.
55. **★★** Finitely many points in the plane, at least three, are given, with no three collinear. Prove that some three form a triangle whose interior contains none of the other points.
56. **★★** Finitely many points in the plane have distinct pairwise distances. Join each point to its nearest neighbor. Prove that the resulting figure contains no cycle.
57. **★★** An odd number of people stand in a field, and all pairwise distances are distinct. Each person throws a water balloon at the closest other person. Prove that at least one person stays dry.
58. **★** A tree is a finite connected graph with no cycles. Prove that every tree with at least two vertices has at least two vertices of degree 1.
59. **★★** In any finite connected network of points and edges, suppose every point is connected to at least two other points. Prove that the network contains a closed loop.
60. **★★** In a round-robin tournament (every pair of players plays exactly once) with no draws, prove that there is a player A such that every other player either lost to A , or lost to someone who lost to A .
61. **★★** In a round-robin tournament with no draws, prove that the players can be lined up so that each player beat the next one.
62. **★** How many ways are there to choose a k -person committee from n people and then choose a chair from the committee? Count the same choices in two ways to prove $k \binom{n}{k} = n \binom{n-1}{k-1}$.
63. **★★** Start with one pile of 100 stones. In each step, split a pile into two smaller piles of sizes A and B , and write AB on the board. Continue until there are 100 piles of size 1. Prove that the total sum written on the board is always the same, and find it.
64. **★★** Twenty clubs each have exactly 4 members, and any two clubs have at most one member in common. Prove that there are at least 16 students.
65. **★★** In a round-robin tournament with n players and no draws, let w_i and ℓ_i be the number of wins and losses of player i . Prove that $\sum_i w_i^2 = \sum_i \ell_i^2$.
66. **★★** Let $d(k)$ be the number of positive divisors of k . Prove that $\sum_{k=1}^n d(k) = \sum_{i=1}^n \lfloor \frac{n}{i} \rfloor$.
67. **★★** Let S be a finite set with $n \geq 1$ elements. Prove that the number of subsets of S with even size equals the number of subsets of S with odd size.
68. **★** Prove that $\{1, \dots, n\}$ has as many subsets of size k as subsets of size $n - k$.
69. **★★** How many k -element subsets of $\{1, \dots, n\}$ contain no two consecutive integers?

70. *** Give a bijection between domino tilings of a $2 \times n$ rectangle and subsets of $\{1, \dots, n-1\}$ with no consecutive elements. Hence find the number of tilings.
71. *** A partition of n is a way of writing n as a sum of positive integers, where order does not matter. Prove that the number of partitions of n into at most k parts equals the number of partitions of n whose largest part is at most k .
72. ** Prove that there exist two powers of 2 that differ by a multiple of 7777.
73. ** Prove that there exists a power of 3 whose last four digits are 0001.
74. ** Prove that in the Fibonacci sequence, there exists a number ending in at least four zeros.
75. *** An 8×8 chessboard has one black and one white square removed. Prove that the remaining squares can be tiled by dominoes.
76. *** A $2^n \times 2^n$ board has one square removed. Prove that the remaining board can be tiled by L-shaped trominoes (each covers three squares of a 2×2 block).
77. *** You are given 15 line segments, each of integer length between 1 and 300. Prove that three of them can form a triangle.
78. *** A group of people is split into two teams, and friendships exist only between people on different teams. Prove that the friendship network contains no cycle of odd length. Conversely, prove that if a friendship network has no cycle of odd length, the people can be split into two teams in this way.
79. *** The real numbers $a_1, a_2, \dots, a_n$, whose sum is 0, are written around a circle. Prove that one can choose a starting point such that all partial sums are nonnegative.
80. *** A cop and a robber are on vertices of a finite tree (a connected graph with no cycles). They move alternately, the cop first; each may move along one edge or stay put. The cop catches the robber when both are on the same vertex. Prove that the cop can catch the robber.
81. *** A room contains 100 politicians. Some pairs are enemies. Prove that the politicians can be divided into two rooms such that every politician has at most as many enemies in their own room as in the other room.
82. *** In every friendship network with n people and m friendships, prove that the people can be divided into two groups so that at least $m/2$ friendships go between the two groups.
83. *** There are n red points and n blue points in the plane, with no three points collinear. Prove that the red and blue points can be paired by n non-intersecting segments, each joining one red point to one blue point.
84. *** In a school of 200 students, every student has exactly 100 friends. Prove that two students have at least 50 common friends.
85. *** In a group of $2n$ people, where $n \geq 2$, each person knows at least n others. Prove that four people can sit around a table so that each person knows both neighbors.
86. *** $2n$ coins of various values lie in a row. Players alternately take a coin from either end. Prove that the first player can guarantee at least a tie in total value.

87. *** Chomp is played on a $2 \times n$ chocolate bar. The bottom-left square is poisoned. A bite at a square removes that square together with every square that is both at least as high and at least as far to the right. Taking the poisoned square loses. Prove that the first player can win.
88. *** Choose 55 numbers from $1, 2, \dots, 100$. Prove that two differ by 10, but show that two need not differ by 11.
89. *** Determine the smallest positive integer k such that, whenever every square of a $3 \times k$ grid is colored red or blue, there must exist a rectangle whose four corner squares all have the same color.
90. *** Color every square of an 8×8 board red or blue. Prove that there exist two rows and two columns such that the four intersection squares are all the same color.
91. *** A chess grandmaster has 77 days to prepare. He plays at least one game every day and at most 12 games during any consecutive 7 days. Prove that there is a sequence of consecutive days during which he plays exactly 21 games.
92. *** The numbers $1, 2, \dots, 100$ are arranged around a circle in any order. Prove that some three consecutive numbers have sum at least 153.
93. *** Write down any 10 distinct integers in a row. Prove that one can circle 4 of them, keeping their left-to-right order, which are either strictly increasing or strictly decreasing.
94. *** How many paths from $(0, 0)$ to (n, n) , using only steps right and up, never go above the diagonal line $y = x$?
95. **** Prove that $1, 2, \dots, n$ can be arranged in a row so that the average of any two entries never appears between them.
96. **** Prove that the number of ways to write n as a sum of distinct positive integers equals the number of ways to write n as a sum of odd positive integers, where in both cases the order of the summands does not matter.
97. **** Two prisoners may agree on a strategy in advance. After that they cannot communicate. The jailer places a coin on each square of a chessboard, each showing heads or tails in any arrangement he likes, and shows prisoner A the board and one chosen square. Prisoner A must flip exactly one coin and then leaves. Prisoner B then enters, sees the board, and must name the chosen square. Find a strategy that guarantees B names the chosen square correctly, whatever the initial arrangement of the coins and whichever square the jailer chooses.
98. **** Let G be a network with n points, where some pairs of points are joined by edges. Prove that if there are more than $\lfloor n^2/4 \rfloor$ edges, then there must be three points all joined pairwise by edges.
99. **** A finite set of points in the plane has the property that every line through two of the points also passes through a third. Prove that all the points lie on one line.
100. **** A 100×100 array is filled with the numbers $1, 2, \dots, 100$, each appearing exactly 100 times. Prove that some row or column contains at least 10 distinct numbers.