

IMO Training: Sequences

Hints

Hints. Try each problem on your own first. Use a hint only when you are stuck, and then go back to the problem.

Level A

- A1.** Look at the differences between consecutive terms. If they are not constant, look at the differences of the differences, or at the ratios.
- A2.** Compute the first few terms and guess a formula. In (c), look at $a_n + 1$. In (e), keep going until something familiar comes back.
- A3.** Write the number of seats in row n . For the total, write the sum forwards and backwards and add the two.
- A4.** Call the total S and compare S with $2S$.
- A5.** After each fold the thickness doubles. Use $2^{10} \approx 1000$.
- A6.** After n steps, how far is Zeno from the wall?
- A7.** What does “average of its two neighbours” say about the differences $a_{n+1} - a_n$? What does “geometric mean” say about the ratios?
- A8.** Start with $a_1 = 1$, $a_2 = 2$ and compute ten terms. Then do the same with letters a and b . For a shorter period: the smallest period of a sequence divides every other period.
- A9.** What can your first move be? How many steps are left after it?
- A10.** Before the largest disc can move, where must all the other discs be?

Level B

- B1.** A sequence is an arithmetic progression with common difference d exactly when $\frac{a_m - a_n}{m - n} = d$ for all $m \neq n$. Write $\frac{a_{b_m} - a_{b_n}}{m - n}$ as a product of two such fractions.
- B2.** Let $t_n = s_{s_n}$, so that $s_{s_n} = t_{s_n}$. Then use the criterion from the previous hint, and split $\frac{t_{s_n} - t_{s_m}}{n - m}$ into two fractions.
- B3.** Look at the largest of the numbers. Where can it be?
- B4.** Let $J(n)$ be the last person left in a circle of n people. Find $J(2n)$ in terms of $J(n)$: after the first round, who is left, and how do they compare with a circle of n people? Then do the same for $J(2n + 1)$.

- B5.** Write n in binary. What do $2n$ and $2n + 1$ look like?
- B6.** Look at the remainder on division by 5. Then show that large numbers always get smaller.
- B7.** What is a_n modulo a_m when $m < n$? For the sum, show that $a_{n+1} - 1 = a_n(a_n - 1)$ for $n \geq 2$.
- B8.** Induction on “the product is < 2 ” does not work. Prove something stronger that involves the sum $a_1 + \cdots + a_n$.
- B9.** Let $u_n = x^n + \frac{1}{x^n}$. Find a recurrence relation for u_n . For the last digit, look for a repeating pattern.
- B10.** Look at $u_n = (2 + \sqrt{3})^n + (2 - \sqrt{3})^n$. Show that $u_{n+2} = 4u_{n+1} - u_n$, and use $0 < 2 - \sqrt{3} < 1$.
- B11.** Consider $y_n = \frac{1}{x_n}$.
- B12.** Rewrite the rule as $a_{n+2}a_n - a_{n+1}^2 = 2$. What happens when you compare this for n and $n + 1$?
- B13.** Write $a_{n+1} = a_n - 1 + \frac{1}{a_n + 1}$ and control the small extra terms.

Level C

- C1.** Rewrite both inequalities using $d_n = (a_0 + a_1 + \cdots + a_n) - na_n$. How does d_{n+1} compare with d_n ?
- C2.** Try to build a sequence that switches between the two kinds of mean forever. Start with 1, 2, 4.
- C3.** The inequality determines each term. Compute a few terms and look for a linear recurrence.
- C4.** First show $a_{n+1} \geq a_n + 1$. When does equality hold? Can it hold twice in a row?
- C5.** Find how $\frac{s_{2n}}{s_{2n+1}}$ and $\frac{s_{2n+1}}{s_{2n+2}}$ relate to $\frac{s_n}{s_{n+1}}$, and run this backwards.
- C6.** Look at $a_n + 1$. Then use $u = 2 + \sqrt{3}$ and $\frac{1}{u} = 2 - \sqrt{3}$.
- C7.** Solve the relation for x_i in terms of x_{i-1} : at each step there are at most two choices.
- C8.** Find an example for $n = 3$. Then combine the relation with shifted copies of itself, sum around the cycle, and aim for a sum of squares.
- C9.** Let $d_n = s_{n+1} - s_n$, and let m and M be its smallest and largest values (why do they exist?). Write $s_{s_{n+1}} - s_{s_n}$ as a sum of d_n consecutive terms d_k , and note that $s_{s_{n+1}} - s_{s_n} = d_{s_n}$.
- C10.** The terms must be spread out very evenly. Try fractional parts of multiples of an irrational number.
- C11.** How fast does a_n grow? Choose $\ell \leq s$ with $\frac{a_\ell}{\ell}$ as large as possible.