

IMO Training: Sequences

A Problem-Solving Curriculum

Instructions. Compute the first terms, look for a pattern, then prove it: a pattern is only a guess until it is proved. Level A starts from the very beginning; Level C reaches the level of the International Mathematical Olympiad. The stars indicate difficulty.

Level A

A1. ★ Find the next two terms of each sequence.

(a) 5, 8, 11, 14, ...

(c) 1, 3, 6, 10, 15, ...

(b) 3, 6, 12, 24, ...

(d) 2, 3, 5, 8, 13, ...

A2. ★ Find a_{100} for each sequence, if you can.

(a) $a_1 = 2$ and $a_{n+1} = a_n + 3$.

(d) $a_1 = 1$ and $a_{n+1} = a_n + n$.

(b) $a_1 = 3$ and $a_{n+1} = 2a_n$.

(c) $a_1 = 1$ and $a_{n+1} = 2a_n + 1$.

(e) $a_1 = 2$ and $a_{n+1} = \frac{1}{1 - a_n}$.

A3. ★ A theatre has 20 seats in the first row, and each row has 3 more seats than the row in front of it. How many seats are in row 25? How many seats are in the first 25 rows together?

A4. ★ A king promises a sage one grain of rice on the first square of a chessboard, two grains on the second, four on the third, and so on, doubling each time. How many grains does the king owe in total?

A5. ★ A sheet of paper is 0.1 mm thick. It is folded in half 42 times. Is it now thicker than the distance from the Earth to the Moon, about 384 000 km?

A6. ★★ Zeno walks towards a wall that is 1 metre away. First he covers half of the distance, then half of what remains, then half of what remains again, and so on. How far has he walked after n steps? Does he ever reach the wall?

A7. ★★ Each term of a sequence, except the first, is the average of its two neighbours. If $a_1 = 2$ and $a_{101} = 502$, find $a_1 + a_2 + \cdots + a_{101}$. What if instead each term is the geometric mean of its two neighbours, all terms are positive, $a_1 = 2$ and $a_4 = 54$? Find $a_1 + a_2 + \cdots + a_{10}$.

A8. ★★ Let $a_1 = a$ and $a_2 = b$ with $a, b > 0$, and $a_{n+1} = \frac{1 + a_n}{a_{n-1}}$ for $n \geq 2$. Compute some terms. What do you notice? Prove it. For which positive integers a, b are all terms integers? Determine all pairs (a, b) for which the smallest period is less than 5, and give that period.

A9. ★★ You climb a staircase taking either 1 or 2 steps at a time. In how many ways can you climb a staircase of 10 steps?

A10. ★★ Three pegs stand in a row, and n discs of different sizes are stacked on the first peg, largest at the bottom. A move takes the top disc of one peg and puts it on another peg, never on a smaller disc. What is the least number of moves needed to bring all the discs to the third peg?

Level B

- B1. ★★** Let $a_1, a_2, \dots$ and $b_1, b_2, \dots$ be arithmetic progressions of positive integers. Prove that $a_{b_1}, a_{b_2}, \dots$ and $b_{a_1}, b_{a_2}, \dots$ are arithmetic progressions with the same common difference. Are they the same sequence?
- B2. ★★** Let (s_n) be a sequence of positive integers such that (s_{s_n}) and $(s_{s_{s_n}})$ are arithmetic progressions with common differences $p > 0$ and $q > 0$.
 (a) Prove that p divides q and that (s_n) is an arithmetic progression with common difference $\frac{q}{p}$.
 (b) Find all possible pairs (p, q) .
- B3. ★★** Real numbers $a_0, a_1, \dots, a_n$ satisfy $a_0 = a_n = 0$ and $a_{k-1} - 2a_k + a_{k+1} \geq 0$ for $0 < k < n$. Prove that $a_k \leq 0$ for all k .
- B4. ★★** n people stand in a circle, numbered $1, 2, \dots, n$. Going round the circle, every second person leaves: first person 2, then 4, and so on, until one person is left. Who is the last person left?
- B5. ★★** Let $a_1 = 1$, $a_{2n} = a_n$ and $a_{2n+1} = a_n + 1$. Find a_{2026} and the largest of $a_1, a_2, \dots, a_{2026}$. How many $n \leq 1024$ satisfy $a_n = 2$?
- B6. ★★** Start with a positive integer. If it is even, halve it; if it is odd, add 5. Repeat. Prove that the sequence always ends up repeating, and find all the blocks that can repeat.
- B7. ★★** Let $a_1 = 1$ and $a_{n+1} = 1 + a_1 a_2 \cdots a_n$. Prove that any two terms are coprime, and find a formula for $\frac{1}{a_1} + \frac{1}{a_2} + \cdots + \frac{1}{a_n}$.
- B8. ★★** Positive numbers $a_1, \dots, a_n$ satisfy $a_1 + \cdots + a_n \leq \frac{1}{2}$. Prove that $(1+a_1)(1+a_2) \cdots (1+a_n) < 2$.
 [Tournament of Towns 2008, Fall, Junior A5]
- B9. ★★** Let $x + \frac{1}{x} = 3$. Prove that $x^n + \frac{1}{x^n}$ is an integer for every positive integer n , and find its last digit for $n = 2026$.
- B10. ★★★** Prove that $\lfloor (2 + \sqrt{3})^n \rfloor$ is odd for every positive integer n .
- B11. ★★★** Find all real numbers x_0, x_1 for which the sequence $x_{n+1} = \frac{x_{n-1}x_n}{3x_{n-1} - 2x_n}$ contains infinitely many positive integers.
- B12. ★★★** Let $a_1 = a_2 = 1$ and $a_{n+2} = \frac{a_{n+1}^2 + 2}{a_n}$. Prove that every term is an integer.
- B13. ★★★** Let $a_0 = 1994$ and $a_{n+1} = \frac{a_n^2}{a_n + 1}$. Prove that $\lfloor a_n \rfloor = 1994 - n$ for $0 \leq n \leq 998$.
 [ISL 1994 A1]

Level C

- C1. ***** Let $a_0 < a_1 < a_2 < \dots$ be an infinite sequence of positive integers. Prove that there is exactly one integer $n \geq 1$ such that

$$a_n < \frac{a_0 + a_1 + \dots + a_n}{n} \leq a_{n+1}.$$

[IMO 2014/1]

- C2. ***** An infinite sequence of distinct positive integers is given. Each term except the first is either the arithmetic mean or the geometric mean of its two neighbours. Must it be true that from some point on, all terms are arithmetic means, or all terms are geometric means?

[Tournament of Towns 2009, Spring, Senior A2]

- C3. ***** A sequence of integers satisfies $a_1 = 2$, $a_2 = 7$ and $-\frac{1}{2} < a_{n+1} - \frac{a_n^2}{a_{n-1}} \leq \frac{1}{2}$ for $n \geq 2$. Prove that a_n is odd for every $n > 1$.

[ISL 1988 #28]

- C4. ***** Let $x_1, \dots, x_{2023}$ be pairwise different positive reals such that

$$a_n = \sqrt{(x_1 + x_2 + \dots + x_n) \left(\frac{1}{x_1} + \frac{1}{x_2} + \dots + \frac{1}{x_n} \right)}$$

is an integer for every $n = 1, \dots, 2023$. Prove that $a_{2023} \geq 3034$.

[IMO 2023/4]

- C5. ***** Let $s_1 = 1$, $s_{2n} = s_n$ and $s_{2n+1} = s_n + s_{n+1}$. Prove that every positive rational number appears exactly once among the fractions $\frac{s_n}{s_{n+1}}$.

- C6. ***** Let $a_0 = 1$, $a_1 = 6$ and $a_{n+1} = 4a_n - a_{n-1} + 2$. Prove that a_{2^k-1} has at least three distinct prime factors for every $k \geq 3$.

[adapted from Balkan MO Shortlist 2014 A6]

- C7. ***** Find the largest x_0 for which there exist positive reals $x_0, x_1, \dots, x_{1995}$ with $x_{1995} = x_0$ and

$$x_{i-1} + \frac{2}{x_{i-1}} = 2x_i + \frac{1}{x_i} \quad (1 \leq i \leq 1995).$$

[IMO 1995/4]

- C8. ***** Find all integers $n \geq 3$ for which there exist real numbers $a_1, \dots, a_{n+2}$ with $a_{n+1} = a_1$, $a_{n+2} = a_2$ and $a_i a_{i+1} + 1 = a_{i+2}$ for $i = 1, \dots, n$.

[IMO 2018/2]

- C9. ***** Let $s_1, s_2, s_3, \dots$ be a strictly increasing sequence of positive integers such that the subsequences $s_{s_1}, s_{s_2}, s_{s_3}, \dots$ and $s_{s_1+1}, s_{s_2+1}, s_{s_3+1}, \dots$ are both arithmetic progressions. Prove that $s_1, s_2, s_3, \dots$ is itself an arithmetic progression.

[IMO 2009/3]

- C10. ****** Let $a > 1$ be a real number. Construct a bounded infinite sequence $x_0, x_1, x_2, \dots$ of real numbers such that $|x_i - x_j| \cdot |i - j|^a \geq 1$ for all $i \neq j$.

[IMO 1991/6]

- C11. ******* Let $a_1, a_2, \dots$ be positive real numbers and s a positive integer such that $a_n = \max\{a_k + a_{n-k} : 1 \leq k \leq n-1\}$ for all $n > s$. Prove that there are positive integers $\ell \leq s$ and N such that $a_n = a_\ell + a_{n-\ell}$ for all $n \geq N$.

[IMO 2010/6]